4 次の問に答えよ．

（1） 等式 $(\tan\theta)' = \dfrac{1}{\cos^2\theta}$ を示せ．また，定積分 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta$ の値を求めよ．

（2） 等式

$$\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta} = \frac{2}{\cos\theta}$$

を示せ．また，定積分 $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta$ の値を求めよ．

（3） 定積分 $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta$ の値を求めよ． （23 佐賀大・医）

4 数学III【定積分】標準

▶解答◀ （**1**） $(\tan\theta)' = \left(\dfrac{\sin\theta}{\cos\theta}\right)'$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

また

$$\left(\frac{\cos\theta}{\sin\theta}\right)' = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta}$$

より

$$\frac{1}{\sin^2\theta} = \left(-\frac{\cos\theta}{\sin\theta}\right)'$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta = \left[-\frac{\cos\theta}{\sin\theta}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \mathbf{1}$$

（**2**） $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{\cos\theta}{1-\sin\theta} = \dfrac{2\cos\theta}{1-\sin^2\theta} = \dfrac{2}{\cos\theta}$

$I_1 = \displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta$ とおく．

$$I_1 = \int_0^{\frac{\pi}{6}} \frac{1}{2}\left(\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left\{\frac{(1+\sin\theta)'}{1+\sin\theta} - \frac{(1-\sin\theta)'}{1-\sin\theta}\right\} d\theta$$

$$= \frac{1}{2}\Big[\log|1+\sin\theta| - \log|1-\sin\theta|\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left\{\log\left(1+\frac{1}{2}\right) - \log\left(1-\frac{1}{2}\right)\right\}$$

$$= \mathbf{\frac{1}{2}\log 3}$$

（**3**） $I_3 = \displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta$ とおく．

ここで

$$I_1 = \int_0^{\frac{\pi}{6}} \frac{(\sin\theta)'}{\cos^2\theta}\,d\theta$$

$$= \left[\frac{\sin\theta}{\cos^2\theta}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin\theta\left(\frac{1}{\cos^2\theta}\right)' d\theta$$

$$= \frac{\frac{1}{2}}{\frac{3}{4}} - \int_0^{\frac{\pi}{6}} \sin\theta\cdot\frac{2\sin\theta}{\cos^3\theta}\,d\theta$$

$$= \frac{2}{3} - 2\int_0^{\frac{\pi}{6}} \frac{1-\cos^2\theta}{\cos^3\theta}\,d\theta$$

$$= \frac{2}{3} - 2I_3 + 2I_1$$

$$I_1 = \frac{2}{3} - 2I_3 + 2I_1$$

$$I_3 = \frac{1}{3} + \frac{1}{2}I_1 = \mathbf{\frac{1}{3} + \frac{1}{4}\log 3}$$

◆別解◆ （**3**） $I_3 = \displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta$

$$= \int_0^{\frac{\pi}{6}} (\tan\theta)'\cdot\frac{1}{\cos\theta}\,d\theta$$

$$= \left[\tan\theta\cdot\frac{1}{\cos\theta}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \tan\theta\cdot\left(\frac{1}{\cos\theta}\right)' d\theta$$

$$= \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}} - \int_0^{\frac{\pi}{6}} \tan\theta\cdot\frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{2}{3} - \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos^3\theta}\,d\theta$$

$$= \frac{2}{3} - \int_0^{\frac{\pi}{6}} \frac{1-\cos^2\theta}{\cos^3\theta}\,d\theta$$

$$= \frac{2}{3} - I_3 + I_1$$

$$I_3 = \frac{2}{3} - I_3 + I_1$$

$$I_3 = \frac{1}{3} + \frac{1}{2}I_1 = \mathbf{\frac{1}{3} + \frac{1}{4}\log 3}$$